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Integral cohomology of classical groups over a finite field

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Abstract

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Let G be one of the groups GL_n, SL_n, U_n, SU_n or Sp_{2n} over a finite field of characteristic p. We calculate the order of the integral Chern classes, obtained by Brauer lifting. From the cellular cochains of suitable Eilenberg–Mac Lane spaces, we construct a complex. Using a description of the mod l cohomology of G (l prime $\neq p$), we prove that the homology of this complex is the integral cohomology of G, away the p-torsion.

Quillen gives the mod l cohomology of GL_n and Fiedorowicz and Priddy give this cohomology for the groups U_n and Sp_{2n}. We compute this cohomology for the special groups SL_n and SU_n.

Introduction

Let $\mathbb{F}_q$ be a finite field with q elements of characteristic p. The purpose of this paper is to describe the integral cohomology, away the p-torsion, of the following groups of matrices (we refer to [5] for the definitions and further details):

the general linear group $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{F}_q)$ and the special linear group $\mathrm{SL}_n = \mathrm{SL}_n(\mathbb{F}_q)$ over $\mathbb{F}_q$,

the unitary group $U_n = U_n(\mathbb{F}_{q^2})$ and the special unitary group $\mathrm{SU}_n = \mathrm{SU}_n(\mathbb{F}_{q^2})$ over $\mathbb{F}_{q^2}$ and

the symplectic group $\mathrm{Sp}_{2n} = \mathrm{Sp}_{2n}(\mathbb{F}_q)$ ($\subset \mathrm{GL}_{2n}$).

Let G denote one of these groups and BG be the associated classifying space. For each integral Chern class $\tilde{c}_i \in H^{2i}(BG; \mathbb{Z})$, obtained by Brauer lifting, we calculate its order s_i and then we choose a subcomplex C_i of the cellular cochain of the Eilenberg–Mac Lane space $K(\mathbb{Z}/s_i, 2i-1)$ such that H^*C_i is the polyno-

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mial subalgebra $\mathbb{Z}[\tilde{x}_i]/(s_i\tilde{x}_i)$ of $H^*(K(\mathbb{Z}/s_i, 2i-1); \mathbb{Z})$, where $\tilde{x}_i$ is a generator of $H^{2i}(K(\mathbb{Z}/s_i, 2i-1); \mathbb{Z}) \simeq \mathbb{Z}/s_i$. Our main result is the following:

Theorem. *$\tilde{H}^*(\bigotimes_{i\in I} C_i)$ is isomorphic to $\tilde{H}^*(BG; \mathbb{Z}[\frac{1}{p}])$ as abelian groups, where I depends on G.*

In [9], Huebschmann gives a description of that cohomology in the stable case, i.e. for $G_\infty = \varinjlim G_n$.

Our proof uses a description of the mod l cohomology of G, for each prime l distinct from p, with generators the Chern classes and other classes e_i, $i \in \mathbb{N}$, considered first by Quillen in [13]. The first part of the paper gives this description. For GL_n this is done in [13]. For U_n and Sp_{2n} we use the complete description of this cohomology given by Fiedorowicz and Priddy in [5].

For SL_n we have two cases. If l divides the order of GL_1 we study the multiplicative structure of the Lyndon–Hochschild–Serre spectral sequence associated to the exact sequence $1 \to \mathrm{SL}_n \to \mathrm{GL}_n \xrightarrow{\det} \mathrm{GL}_1 \to 1$. In particular, we show that the spectral sequence collapses and the GL_1-action on $H^*(\mathrm{SL}_n; \mathbb{Z}/l)$ is trivial. If l does not divide the order of GL_1 we use some basic tools of cohomology of groups and we adapt the detection theorems of Quillen (cf. [12]) to our situation.

It turns out that the same method works for SU_n.

The orthogonal groups are not treated here. This case seems to be somewhat different from the others, especially about the 2-torsion (cf. [5, I.7]). Moreover, the method used to compute the cohomology of the symplectic groups does not work. Indeed the homomorphism induced by the inclusion $O_2(\mathbb{F}_q) \to \mathrm{GL}_2(\mathbb{F}_q)$ in mod 2 cohomology is not surjective (cf. Theorem 1.3 and [5, IV.4 and IV.8]). On the other hand there are different kinds of orthogonal groups and it would be better to treat them in another paper.

1. Mod l cohomology

Let k be an integer and r the smallest integer ≥ 1 such that $k^r \equiv 1 \pmod{l}$. For any space or group X we note $H^*X = H^*(X; \mathbb{Z}/l)$. By $P[x_1, x_2, \ldots]$ (resp. $\Lambda[x_1, x_2, \ldots]$) we mean the polynomial ring (resp. exterior algebra) over $\mathbb{Z}/l$ with the generators $x_1, x_2, \ldots$. We consider the classical fibration

$$JU(k) \xrightarrow{\phi} BU \xrightarrow{1-\psi^k} BU\,, \tag{1}$$

where $1-\psi^k$ is the application representing the Adams operation in K-theory. We know that $H^*BU = P[c_1', c_2', \ldots]$ where $c_i' \in H^{2i}BU$ is the ith universal Chern class. We put $c_i = \phi^* c_i' \in H^{2i}JU(k)$. Quillen shows in [13] that $c_i = 0$ for $i \not\equiv 0 \pmod{r}$ and for $i \equiv 0 \pmod{r}$ he constructs classes $e_i \in H^{2i-1}JU(k)$. The case where $l \neq 2$ or if $l = 2$ then $k \equiv 1 \pmod 4$ is called the typical one and the case where $l = 2$ and $k \equiv 3 \pmod 4$ is the exceptional one.

Theorem 1.1 (Quillen). *The monomials*

$$c_r^{\alpha_1} c_{2r}^{\alpha_2} \cdots e_r^{\beta_1} e_{2r}^{\beta_2} \cdots$$

with $\alpha_i \in \mathbb{N}$ and $\beta_i \in \{0, 1\}$ form a basis for $H^*JU(k)$. In the typical case $(e_{jr})^2 = 0$ and there is an algebra isomorphism

$$H^*JU(k) \simeq P[c_r, c_{2r}, \ldots] \otimes \Lambda[e_r, e_{2r}, \ldots] .$$

In the exceptional case,

$$e_j^2 = \sum_{0 \leq i < j} c_i c_{2j-1-i} . \qquad \square$$

Let G_n denote GL_n or U_n. Let $k = q$ for $G_n = \mathrm{GL}_n(\mathbb{F}_q)$ and $k = -q$ for $G_n = U_n(\mathbb{F}_{q^2})$. Let G_∞ be the direct limit $\varinjlim G_n$. The space $JU(k)$ has the homotopy type of the space BG_∞^+ obtained by performing the '+' construction on the classifying space BG_n (cf. [5, 7, 13]). We consider the canonical map

$$\varepsilon : BG_n \to BG_\infty \to BG_\infty^+ \sim JU(k) .$$

We write also c_i and e_i for the image of these classes under ε^* or the homomorphisms induced by canonical inclusions of groups.

Theorem 1.2. *The algebra homomorphism $\varepsilon^* : H^*JU(k) \to H^*BG_n$ is surjective. The kernel is the ideal generated by e_{jr} and c_j for $jr > n$. In particular, the monomials*

$$c_r^{\alpha_1} c_{2r}^{\alpha_2} \cdots c_{mr}^{\alpha_m} e_r^{\beta_1} e_{2r}^{\beta_2} \cdots e_{mr}^{\beta_m} ,$$

with $\alpha_i \in \mathbb{N}$, $\beta_i \in \{0, 1\}$ and $m = [n/r]$, form a basis of H^*G_n.

Let SG_n denote the groups SL_n or SU_n.

Theorem 1.3. *The algebra homomorphism $H^*G_n \to H^*\mathrm{SG}_n$, induced by the inclusion, is surjective. The kernel is the ideal generated by e_1 and c_1. For $r \geq 2$ it is an isomorphism.*

Let d be the smallest integer such that $q^{2d} \equiv 1 \pmod l$.

Theorem 1.4. *The algebra homomorphism $H^*\mathrm{GL}_{2n} \to H^*\mathrm{Sp}_{2n}$, induced by the* *inclusion, is surjective. If $r = 2d$, it is an isomorphism. If $r = d$, the kernel is the ideal generated by $e_{(2j+1)r}$ and $c_{(2j+1)r}$, for $j \in \mathbb{N}$.*

In both cases, we have

$$H^*\mathrm{Sp}_{2n} = P[c_{2d}, c_{4d}, \ldots, c_{2dm}] \otimes \Lambda[e_{2d}, e_{4d}, \ldots, e_{2dm}]$$

for $1 \leq n \leq \infty$, where $m = [n/2d]$.

2. Integral cohomology away the p-torsion

Let $\tilde{c}'_i \in H^{2i}(BU; \mathbb{Z})$ be the ith integral universal Chern class. We consider the fibration (1) and the canonical map $\varepsilon : BG_n \to BG_\infty^+ \sim JU(k)$. We write $\tilde{c}_i$ for the image of $\tilde{c}'_i$ under the homomorphisms Φ^*, $\varepsilon^* \circ \Phi^*$ and those induced by inclusions of groups.

Theorem 2.1. *We have*:

(1) *For G_n and $1 \leq i \leq n$, the order of $\tilde{c}_i$ is $|k^i - 1|$.*

(2) *For SG_n and $2 \leq i \leq n$, the order of $\tilde{c}_i$ is $|k^i - 1|$ and $\tilde{c}_1$ is zero.*

(3) *For $i > n$, $\tilde{c}_i = 0$.*

Let G denote G_n or SG_n and let I be the set of the i's such that $1 \leq i \leq n$ for G_n and $2 \leq i \leq n$ for SG_n. For each $i \in I$, let s_i denote the order of $\tilde{c}_i$ in $H^{2i}(BG; \mathbb{Z})$.

We consider the canonical fibration of Eilenberg–Mac Lane spaces

$$K(\mathbb{Z}/s_i, 2i-1) \xrightarrow{\beta} K(\mathbb{Z}, 2i) \xrightarrow{s_i} K(\mathbb{Z}, 2i)\,, \tag{2}$$

where s_i is induced by the multiplication by s_i on $\mathbb{Z}$.

We denote by the same letter a cohomology class and its corresponding map under the usual isomorphism $[X, K(R, m)] \simeq H^m(X; R)$.

In [13], Quillen builds a class $\tilde{e}_i \in H^{2i}(BG; \mathbb{Z}/s_i)$ such that

$$\beta \tilde{e}_i = \tilde{c}_i\,, \tag{3}$$

where β denote the Bockstein homomorphism associated to the coefficient exact sequence

$$O \to \mathbb{Z} \xrightarrow{s_i} \mathbb{Z} \to \mathbb{Z}/s_i \to O\,.$$

Thus

$$\gamma_i = \tilde{e}_i : BG \to K(\mathbb{Z}/s_i, 2i-1)$$

is a lifting of $\tilde{c}_i$. We put $K_i = K(\mathbb{Z}/s_i, 2i-1)$.

The generator $\tilde{x}_i$ of $H^{2i}(K_i;\mathbb{Z}) \simeq \mathbb{Z}/s_i$ generates a polynomial subalgebra $P_i = \mathbb{Z}[\tilde{x}_i]/(s_i\tilde{x}_i)$ of $H^*(K_i;\mathbb{Z})$ and P_i is a direct summand (cf. [4]).

Lemma 2.2. *We can choose $\tilde{x}_i$ such that $\gamma_i^*\tilde{x}_i = \tilde{c}_i$.*

Proof. Let $\iota \in H^{2i}(K(\mathbb{Z},2i);\mathbb{Z})$ be the fundamental class. The Serre cohomology sequence of (2) shows that $\beta^*\iota$ is a generator of $H^{2i}(K_i;\mathbb{Z})$. Then by (3), $\gamma_i^*(\beta^*\iota) = \beta\gamma_i = \beta\tilde{e}_i = \tilde{c}_i$. □

We denote by $(C(X), d)$ the cellular cochain complex of a space X. We chose a cocycle $z_i \in C^{2i}(K_i)$ which represents $\tilde{x}_i$ and $b_i \in C^{2i-1}(K_i)$ such that $db_i = s_i z_i$. Let C_i be the subcomplex of $C(K_i)$ given by

$$C_i^{2im} = \mathbb{Z}z_i^m\,, \quad c_i^{2im-1} = \mathbb{Z}z_i^{m-1} \cup b_i\,, \quad C_i^j = 0 \text{ otherwise},$$

where $\cup$ is the cup product of the cochains.

Remark. The homomorphism $H^*C_i \to H^*(K_i;\mathbb{Z})$, induced by the inclusion, is injective with image P_i.

Let $g : BG \to \prod_{i\in I} K_i$ be the map given by the γ_i's and

$$h : \bigotimes_{i\in I} C_i \to \bigotimes_{i\in I} C(K_i) \simeq C\Big(\prod_{i\in I} K_i\Big) \xrightarrow{g^*} C(BG)$$

be the homomorphism of complexes obtained by the Eilenberg–Zilber equivalence.

Theorem 2.3. *The homomorphism*

$$h^* : \tilde{H}^*(\bigotimes_{i\in I} C_i) \to \tilde{H}^*(BG;\mathbb{Z}[\tfrac{1}{p}])$$

is an isomorphism of abelian groups.

We consider the canonical homomorphism

$$f : \bigotimes_{i=1}^{n} C_{2i} \to \bigotimes_{i=1}^{2n} C_i \xrightarrow{h} C(B\mathrm{GL}_{2n}) \to C(B\mathrm{Sp}_{2n})\,.$$

Theorem 2.4. *The homomorphism*

$$f^* : \tilde{H}^*(\bigotimes_{i=1}^{n} C_{2i}) \to \tilde{H}^*(B\mathrm{Sp}_{2n};\mathbb{Z}[\tfrac{1}{p}])$$

is an isomorphism of abelian groups.

Corollary 2.5. *The order of the integral Chern class $\tilde{c}_{2i} \in H^{2i}(B\mathrm{Sp}_{2n}; \mathbb{Z})$ is $q^{2i} - 1$ for $1 \leq i \leq n$. The other classes are zero.* □

Remarks. (1) The theorems remain true in the stable case, i.e. for $n = \infty$. So they describe $\tilde{H}^*(BG_\infty; \mathbb{Z})$, $\tilde{H}^*(B\mathrm{SG}_\infty; \mathbb{Z})$ and $\tilde{H}^*(B\mathrm{Sp}_\infty; \mathbb{Z})$ since these groups have no p-torsion (cf. [7, 13] and (11)).

(2) Let R and R' be the subalgebras generated by the x_i's and $\tilde{c}_i$'s respectively. Then $h^* : R \to R'$ is an isomorphism of algebras.

(3) The groups $H^k(\bigotimes_{i\in I} C_i)$ are easily computed by the Künneth formula.

Example. Let $G = \mathrm{SL}_4(\mathbb{F}_5)$. The mod l cohomology is not trivial only for $l =$ 2,3,13 or 31 ($l \neq 5$). The involved Chern classes are $\tilde{c}_2$, $\tilde{c}_3$ and $\tilde{c}_4$, their orders are 24, 124 and 624 respectively. Then we have $H^{4j}C_2 = \mathbb{Z}/24z_2^j$, $H^{6j}C_3 = \mathbb{Z}/124z_3^j$, $H^{8j}C_4 = \mathbb{Z}/624z_4^j$ and the other groups are zero. By applying the Künneth formula to compute $H^*(C_2 \otimes C_3)$ and $H^*(C_2 \otimes C_3 \otimes C_4)$ successively, we obtain the integral cohomology groups of G (away the 5-torsion):

$$H^4G = \mathbb{Z}/24\,, \quad H^6G = \mathbb{Z}/124\,, \quad H^8G = \mathbb{Z}/24 \otimes \mathbb{Z}/624\,,$$

$$H^9G = \mathbb{Z}/4\,, \quad H^{10}G = \mathbb{Z}/4\,, \quad H^{11}G = \mathbb{Z}/4\,,$$

$$H^{12}G = \mathbb{Z}/4 \otimes \mathbb{Z}/24 \otimes \mathbb{Z}/124\,, \quad \ldots$$

The nonlisted groups are zero.

3. Proof of Theorem 1.2

In [13], Quillen proves the theorem for $G = \mathrm{GL}_n$. For $G = U_n$ we use the results of [5] and [13]. If $n < r$, l does not divide the order of U_n and then the mod l cohomology of U_n is trivial. For $n = r$ the l-Sylow subgroup C of U_r is cyclic and its order is l^a with $a = \nu_l((-q)^r - 1)$. We get a commutative diagram

$$\begin{tikzcd}
BC \arrow[r, "i"] \arrow[d, "Bf"] & BU_r \arrow[rr] & & JU(-q) \arrow[d, "\Phi"] \\
\prod_{i=1}^{r} BC \arrow[r, "Bg"] & \prod_{i=1}^{r} BU(1) \arrow[r] & BU(r) \arrow[r] & BU
\end{tikzcd}$$

where i is induced by inclusion, $f(x) = (x, x^\alpha, x^{\alpha^2}, \ldots, x^{\alpha^{r-1}})$ with $\alpha = q$ if r is even and $\alpha = q^2$ if r is odd, and $g = \prod_{i=1}^{r} \rho$ where $\rho : C \to S^1 = U(1)$ is the canonical representation (cf. [5]). Now we compute $\varepsilon^* i^* c_{jr}$ and $\varepsilon^* i^* e_{jr}$ as in the proof of Proposition 1 in [13]. In mod l cohomology i^* is injective and the Poincaré series of H^*BU_r is $(1 + t^{2r-1})/(1 - t^{2r})$. The result follows from the multiplicative structure of H^*BC.

For $n > r$ the map $BU_\infty \to JU(-q)$ induces an isomorphism in mod l cohomology. By [5] the canonical homomorphism $(U_r)^m \xrightarrow{\oplus} U_n$ (resp. $U_n \to U_\infty$) induces a monomorphism (resp. an epimorphism) in mod l cohomology. The proof for $n = r$ and the same computation as in the proof of Lemma 8 in [13], using the product formula, imply the theorem. $\square$

4. Proof of Theorem 1.3 in the case $r = 1$

The exact sequence

$$1 \to \mathrm{SG}_n \to G_n \xrightarrow{\det} C \to 1 \tag{4}$$

splits and G_n is the semi-direct product $\mathrm{SG}_n \rtimes C$, is the cyclic group G_1. Let Y be the usual free resolution of $\mathbb{Z}$ over $\mathbb{Z}C$ and let X be the standard resolution of $\mathbb{Z}$ over $\mathbb{Z}\mathrm{SG}_n$ (cf. [3, I.5]). By a result of [11], $X \otimes Y$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G_n$ and the cohomology of G_n is the homology of the bigraded differential algebra $B = \mathrm{Hom}_C(Y, \mathrm{Hom}_{\mathrm{SG}_n}(X; \mathbb{Z}/l))$. There is a filtration on B defined by $F^r(B^n) = \bigoplus_{k=r}^n B_{k,n-k}$ where $B_{p,q} = \mathrm{Hom}_C(Y_p, \mathrm{Hom}_{\mathrm{SG}_n}(X_q; \mathbb{Z}/l))$ and $B^n = \bigoplus_{p+q=n} B_{p,q}$. The corresponding spectral sequence E is the Lyndon–Hochschild–Serre spectral sequence of (4) (cf. [3, VII] and [11]). For $x \in H^*G_n$ we note $\bar{x}$ the image of x in $E_\infty \simeq \mathrm{Gr}H^*G_n$.

Proposition 4.1. *For this spectral sequence we have:*

(1) *The filtration degree of the monomial $c_1^{\alpha_1}c_2^{\alpha_2}\cdots c_n^{\alpha_n}e_1^{\beta_1}e_2^{\beta_2}\cdots e_n^{\beta_n}$ is $2\alpha_1 + \beta_1$ and the monomials $\bar{c}_1^{\alpha_1}\bar{c}_2^{\alpha_2}\cdots\bar{c}_n^{\alpha_n}\bar{e}_1^{\beta_1}\bar{e}_2^{\beta_2}\cdots\bar{e}_n^{\beta_n}$ of degree $p + q = 2(\alpha_1 + \cdots + \alpha_n) + \beta_1 + \cdots + \beta_n$ with $p = 2\alpha_1 + \beta_1$ form a basis of $E_\infty^{p,q}$.*

(2) *$E_\infty^{p,q} = E_2^{p,q}$ for all p and q.*

(3) *The C-action on $H^*\mathrm{SG}_n$, induced by the C-action on SG_n, is trivial.*

Now, since $E_2^{p,q} \simeq H^p(C; H^q\mathrm{SG}_n)$, we have $E_\infty^{0,q} \simeq H^q\mathrm{SG}_n$ and Theorem 1.3 follows from Proposition 4.1.

Proof of Proposition 4.1. We proceed by induction on q. Since the sequence (4) splits, $\det^* : H^*C \to H^*G_n$ is injective and then $\mathrm{Im}\det^* = E_\infty^{*,0} \subset H^*G_n$ is isomorphic to H^*G_1. Now the C-action on $H^0\mathrm{SG}_n$ is trivial and $E_\infty^{*,0}$ is a quotient of $E_2^{*,0} \simeq H^*(G_1; H^0\mathrm{SG}_n) = H^*G_1$. This proves the proposition for $q = 0$. Suppose the proposition is true for $s \leq q - 1$ and write $b(m)$ for a monomial of the form $c_2^{\alpha_2}\cdots c_n^{\alpha_n}e_2^{\beta_2}\cdots e_n^{\beta_n}$ in H^mG_n. From $H^qG_n \simeq \bigoplus_{r+s=q} E_\infty^{r,s}$ and the multiplicative structure of E_∞ we deduce:

> The filtration degree of the monomials $c_1^{\alpha_1}e_1^{\beta_1}b(q)$ is $2\alpha_1 + \beta_1$ and their images $\bar{c}_1^{\alpha_1}\bar{e}_1^{\beta_1}\bar{b}(q)$ in $E_\infty^{2\alpha_1+\beta_1,q}$ are linearly independent; moreover, the $\bar{b}(q)$ form a basis of $E_\infty^{0,q}$. In particular, the multiplication by $\bar{c}_1^{\alpha_1}\bar{e}_1^{\beta_1}$ is injective in E_∞. (5)

The differentials $d_r^{p,q}$ are zero for $r \geq 2$. In particular,

$$E_\infty^{0,q} = E_2^{0,q} \quad \text{and} \quad E_\infty^{1,q} = E_2^{1,q}. \tag{6}$$

Lemma 4.2. *The C-action on $H^q \mathrm{SG}_n$ is trivial.*

Hence by the universal coefficient theorem we have $E_2^{p,q} = E_2^{p,0} \otimes E_2^{0,q}$. From (5) and (6) it follows that $E_\infty^{p,q} = E_2^{p,q}$ for all p. This proves Proposition 4.1. □

Proof of Lemma 4.2. We have an isomorphism of algebras

$$E_1^{*,q} \simeq \mathrm{Hom}_C(Y_*; H^q\mathrm{SG}_n). \tag{7}$$

Since $Y_p = \mathbb{Z}C$ for all p, we get a commutative diagram.

$$\begin{array}{ccccc} E_1^{0,q} & \xrightarrow{d_1^{0,q}} & E_1^{1,q} & \xrightarrow{d_1^{1,q}} & E_1^{2,q} \\ \wr\| & & \wr\| & & \wr\| \\ H^q\mathrm{SG}_n & \xrightarrow{g-1} & H^q\mathrm{SG}_n & \xrightarrow{N} & H^q\mathrm{SG}_n \end{array} \tag{8}$$

where g is a generator of C and $N = 1 + g + g^2 + \cdots + g^t$ with $t = |C| - 1$.

Claim 4.3. *The multiplication $E_1^{0,q} \otimes E_1^{1,0} \to E_1^{1,q}$ is an isomorphism.*

Proof. By an argument of dimension it suffices to prove the surjectivity. Since $E_1^{1,0} = E_2^{1,0}$ we can consider the class $e_1 \in E_2^{1,0} = H^1C$ as the cocycle in $E_1^{1,0}$ defined by $e_1(1) = 1 \in H^0\mathrm{SG}_n = \mathbb{Z}/l$. Using the usual diagonal approximation associated with the resolution Y (cf. [3, V.1]) we get

$$(-1)^q(\varphi \cup e_1) = \varphi \quad \text{for } \varphi \in \mathrm{Hom}_C(Y; H^q\mathrm{SG}_n). \tag{9}$$

The claim follows now from (7). □

Thus we can write every element in $E_1^{1,q}$ as $\varphi \cdot e_1$ for an unique $\varphi \in E_1^{0,q}$. Now if $x \in H^q\mathrm{SG}_n$ is not fixed under the C-action then there exists a $\varphi \in E_1^{0,q}$ such that

$$d_1^{0,q}x = \varphi \cdot e_1 \neq 0 \tag{10}$$

and then $d_1^{1,q}d_1^{0,q}x = (d_1^{0,q}\varphi) \cdot e_1 = 0$. From (8) and (9) one sees that φ is fixed in $H^q\mathrm{SG}_n$. Since $(H^q\mathrm{SG}_n)^C \simeq E_2^{0,q} \subset E_1^{0,q}$ and $E_\infty^{0,q} = E_2^{0,q}$ by (6), $\bar{\varphi}$ is non-zero in E_∞. Since $\bar{e}_1$ is also non-zero in E_∞, $\bar{\varphi} \cdot \bar{e}_1$ is non-zero by (5). This contradicts (10) and proves Lemma 4.2. □

Remark. Instead of using the complex B we can obtain a similar proof by taking the cellular cochain complex of $BC = K(C, 1)$ with value in the system of local coefficients $H^q\mathrm{SG}_n$.

5. Proof of Theorem 1.3 (continued)

In this section we prove Theorem 1.3 for $\mathrm{SG}_n = \mathrm{SL}_n$ if $r > 1$ and for $\mathrm{SG}_n = \mathrm{SU}_n$ if r is odd >1. The idea is to follow the method of [13, Corollary of Theorem 3].

Since $r > 1$, l is odd. Let t be q for SL_n and q^2 for SU_n. The l-Sylow subgroup C of SG_r is cyclic of order l^a with $a = \nu_l(t^r - 1)$ and is given as follows.

For SL_r we consider $\mathbb{F}_{q^r}$ as an r-dimensional vector space over $\mathbb{F}_q$ and we get a monomorphism $h : \mathbb{F}_{q^r}^* \to \mathrm{GL}_r(\mathbb{F}_q)$ given by $h(a)x = ax$.

For SU_r we consider the standard Hermitian space $\mathbb{F}_{q^{2r}}$ (with form H) as an r-dimensional Hermitian space over $\mathbb{F}_{q^2}$ with the form $\mathrm{tr} \circ H$, where $\mathrm{tr} : \mathbb{F}_{q^{2r}} \to \mathbb{F}_{q^2}$ is the trace map (cf. [5], note that r odd is required). As above we get a monomorphism $h : U_1(\mathbb{F}_{q^{2r}}) \to U_r(\mathbb{F}_{q^2})$ given by $h(a)x = ax$.

Then C is the p-primary component of $F_{q^r}^*$ for SL_n, resp. $U_1(\mathbb{F}_{q^{2r}})$ for SU_n, and $h(C)$ is contained in SG_r since $r > 1$.

We put $n = mr + e$ with $0 \leq e < r$ and we obtain a monomorphism

$$i : C^m \to \mathrm{SG}_n\,, \qquad x = (x_1, \dots, x_m) \mapsto (h(x_1), \dots, h(x_m), I_e)\,.$$

Proposition 5.1. *The homomorphism $i^* : H^*\mathrm{SG}_n \to H^*C^m$ is injective.*

The Galois group $\pi = \mathrm{Gal}(\mathbb{F}_{t^r}/\mathbb{F}_t)$ is cyclic of order r generated by the Frobenius homomorphism $F : x \mapsto x^t$ and it acts on C. We get then an action of the wreath product $\pi^m \rtimes \Sigma_m$ on C^m, where the symmetric group Σ_m permutes canonically the factors.

Proposition 5.2. *In SG_n the action of any element of $\pi^m \rtimes \Sigma_m$ on C^m is realized by an inner automorphism.*

An inner automorphism induces the identity in cohomology and thus we have a monomorphism

$$i^* : H^*\mathrm{SG}_n \to (H^*C^m)^{\pi^m \rtimes \Sigma_m}\,.$$

Now we consider the canonical homomorphism.

$$j^* : H^*G_n \to H^*\mathrm{SG}_n \xrightarrow{i^*} (H^*C^m)^{\pi^m \rtimes \Sigma_m}$$

and Theorem 1.3 follows from the following proposition.

Proposition 5.3. *The homomorphism $j^* : H^*G_n \to (H^*C^m)^{\pi^m \rtimes \Sigma_m}$ is an isomorphism.*

Proof of Proposition 5.1. Since the index of SG_{mr} in SG_n is relatively prime to l, it suffices to show that C^m detects the mod l cohomology of SU_{mr}.

We consider the embedding $\Sigma_m \to G_{mr}$ such that Σ_m acts by conjugation on the diagonal matrices of m blocks $r \times r$ by permuting these blocks in the obvious way. The index of the subgroup $H_{m,r} = \Sigma_m \cap \mathrm{SG}_{mr}$ of Σ_m $(\subset G_{mr})$ is 1 or 2 and, since l is odd, the wreath product $(\mathrm{SG}_r)^m \rtimes H_{m,r}$ detects the cohomology of SG_{mr}. But the l-Sylow subgroup C of SG_r detects the cohomology of SG_r and the following lemma implies Proposition 5.1. □

Lemma 5.4. *Let G be a subgroup of* SG_r. *Suppose the* mod l *cohomology of G is detected by a subgroup K of G. Then the* mod l *cohomology of the wreath product $G^m \rtimes H_{m,r}$ $(\subset \mathrm{SG}_{mr})$ is detected by K^m.*

Proof. By induction on m. There is nothing to prove for $m = 1$. We have two cases.

(1) If m is relatively prime to l then the index of $(G^{m-1} \rtimes H_{m-1,r}) \times G$ in $G^m \rtimes H_{m,r}$ is relatively prime to l and thus this subgroup detects the cohomology of $G^m \rtimes H_{m,r}$. The lemma follows by induction and the Künneth formula.

(2) If l divides m, we write $m = \bar{m}l$ and we consider the subgroup

$$(G^l \rtimes H_{l,r})^{\bar{m}} \rtimes H_{\bar{m},lr} \simeq G^m \rtimes (H_{l,r}^{\bar{m}} \rtimes H_{\bar{m},lr})$$

of $G^m \rtimes H_{m,r}$. By induction its cohomology is detected by $(K^l)^{\bar{m}} = K^m$. But its index is relatively prime to l and so this subgroup detects the cohomology of $G^m \rtimes H_{m,r}$. This proves Lemma 5.4. □

Proof of Proposition 5.2. We have $\Sigma_m \subset G_{mr} \subset G_n$. Moreover, $F \in G_r$ and for each $x \in C$ we have $F \circ h(x) \circ F^{-1} = h(F(x))$. So, in G_n, $\pi^m \rtimes \Sigma_m$ acts on C^m by inner automorphisms. Now for each $x \in G_n$ there exists b in $\mathbb{F}_{q^r}^*$ (or $U_1(\mathbb{F}_{q^{2r}})$) such that $\det h(b) = \det x^{-1}$, since $\det \circ h$ is surjective. But $\mathbb{F}_{q^r}^*$ and $U_1(\mathbb{F}_{q^{2r}})$ are cyclic and then $x \cdot i(b, 1, \ldots, 1) \in \mathrm{SG}_n$ acts as x on C^m. □

Proof of Proposition 5.3. Quillen proved it for GL_n in [13]. For U_n, we remark that $j : C^m \to \mathrm{SU}_n \to U_n$ factorizes by

$$C^m \to (U_r)^m \to U_n\,.$$

The proof of Theorem 1.2 (or [13, Section 5]) shows that $H^*U_r \to (H^*C)^\pi$ and $H^*U_n \to ((H^*U_r)^{\otimes m})^{\Sigma_m}$ are isomorphisms. □

6. Proof of Theorem 1.4

Let $i : \mathrm{Sp}_{2n} \to \mathrm{GL}_{2n}$ be the inclusion. We write $2n = rs + e = 2dm + v$ with $0 \le e < r$ and $0 \le v < 2d$. For $r = 2d$, l does not divide the index of Sp_{2n} in GL_{2n} and then i^* is injective. We have $m = s$ and by comparing the Poincaré series of $H^*\mathrm{GL}_{2n}$ and $H^*\mathrm{Sp}_{2n}$ we see that i^* is surjective. For $r = d$, we first show that the theorem is true for $n = d$.

Lemma 6.1. $H^*\mathrm{Sp}_{2n} = P[c_{2d}] \otimes \Lambda[e_{2d}]$.

Then we consider the canonical map

$$f : (\mathrm{Sp}_{2d})^m \xrightarrow{j_1} \mathrm{Sp}_{2dm} \xrightarrow{j_2} \mathrm{Sp}_{2n} \to \mathrm{GL}_{2n} \to \mathrm{GL}_\infty .$$

By Lemma 6.1 we can write $H^*(\mathrm{Sp}_{2d})^m = P[x_1, \ldots, x_m] \otimes \Lambda[y_1, \ldots, y_m]$, where x_i (resp. y_i) corresponds to the class c_{2d} (resp. e_{2d}) of the ith factor. Using the notation of Lemma 9 in [13] and the product formula we show by induction on m that for each integer k

$$f^*c_{2dk} = \sigma_k , \quad f^*e_{2dk} = d\sigma_k , \quad f^*c_{(2k+1)d} = f^*e_{(2k+1)d} = 0 .$$

In particular, $f^*(e_{2dk})^2 = 0$ (cf. [13, p. 565]). By Theorem 1.2 the image of $H^*\mathrm{GL}_{2n}$ in $H^*(\mathrm{Sp}_{2d})^m$ is then $P[\sigma_1, \ldots, \sigma_m] \otimes \Lambda[d\sigma_1, \ldots, d\sigma_m]$ and the theorem follows from the fact that j_1^* and j_2^* are monomorphisms (cf. [5]). □

Proof of Lemma 6.1. For $d = 1$, $\mathrm{Sp}_2 = \mathrm{SL}_2$ and the lemma follows from Theorem 1.3 and the fact that $e_2^2 = 0$. If $d > 1$ then l is odd and we consider

$$f : \mathrm{GL}_d \to \mathrm{Sp}_{2d} \to \mathrm{GL}_{2d}$$

defined by

$$\mathrm{GL}_d \xrightarrow{\Delta} \mathrm{GL}_d \times \mathrm{GL}_d \xrightarrow{1 \times g} \mathrm{GL}_d \times \mathrm{GL}_d \xrightarrow{h} \mathrm{GL}_{2d} \xrightarrow{c} \mathrm{GL}_{2d} ,$$

where Δ is the diagonal map, $g(x) = (x^{-1})^t$, $h(x, y) = \mathrm{diag}(x, y)$ and c is the conjugation sending the $2d \times 2d$ matrix $\left(\begin{smallmatrix} 0 & I_d \\ -I_d & 0 \end{smallmatrix}\right)$ to $\mathrm{diag}(A, \ldots, A)$ where $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2$. Now c^* is the identity and $H^*\mathrm{GL}_d = P[c_d] \otimes \Lambda[e_d]$ since $r = d$ and l is odd.

Lemma 6.2. *$f^*c_{2d} = \delta(c_d)^2$ and $f^*e_{2d} = \delta c_d e_d$ with $(\delta, q^d - 1) = 1$.*

Hence f^* maps the subalgebra $P[c_{2d}] \otimes \Lambda[e_{2d}]$ of $H^*\mathrm{GL}_{2d}$ injectively in $H^*\mathrm{GL}_d$ and this subalgebra is isomorphic to $H^*\mathrm{Sp}_{2d}$ by [5]. This proves Lemma 6.1. □

Proof of Lemma 6.2. The integral Chern class $\tilde{c}_d$ generates a direct summand $\mathbb{Z}/q^d - 1$ in the group $H^{2d}(\mathrm{GL}_d; \mathbb{Z})$ (cf. [10] or Theorem 2.3). Hence $g^*\tilde{c}_d = \delta\tilde{c}_d$ with $(\delta, q^d - 1) = 1$. Since $H^{2d-1}(\mathrm{GL}_d; \mathbb{Z})$ is a p-group (cf. same ref.), the Bockstein homomorphism $\beta : H^{2d-1}(\mathrm{GL}_d; \mathbb{Z}/l) \to H^{2d}(\mathrm{GL}_d; \mathbb{Z})$ is injective and the same relation holds for e_d. The lemma follows from the product formula (cf. [13, p. 563]). □

7. Proof of Theorem 1.3 (end)

It remains to prove the case r even for $\mathrm{SG}_n = \mathrm{SU}_n$. We have $r = 2d$ (cf. Section 1 for the notation) and l odd. If $n < r$, l does not divide the order of SU_n and then the mod l cohomology of SU_n is trivial.

For $n = r$ by [5] we have the following monomorphisms:

$$\mathrm{Sp}_{mr}(\mathbb{F}_q) \xrightarrow{g} \mathrm{SU}_{mr}(\mathbb{F}_{q^2}) \xrightarrow{h} U_{mr}(\mathbb{F}_{q^2}) .$$

Since the indices of these subgroups are relatively prime to l, g^* and h^* are injective. By comparing the Poincaré series we see that $g^* \circ h^*$ is then an isomorphism, so g^* and h^* are isomorphisms.

For $n = mr + e$, $0 \leq e < r$, we consider the commutative diagram of inclusions

$$\begin{array}{ccc} \mathrm{SU}_{mr} & \xrightarrow{f} & \mathrm{SU}_n \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ U_{mr} & \xrightarrow{k} & U_n \end{array}$$

We know that h^* and k^* are isomorphisms and by an argument of transfer f^* and g^* are injective. Hence f^* and g^* are isomorphisms. □

8. Proof of Theorem 2.1

We first prove the following result:

Proposition 8.1. *The class $\tilde{c}_i = \Phi^*\tilde{c}'_i$ has order $|k^i - 1|$ in $H^{2i}(JU(k); \mathbb{Z})$. In particular, $\tilde{c}_i$ has no p-torsion.*

Proof. Since ΩBU has the same homotopy type as U, we get the fibration

$$(*) \qquad U \to JU(k) \xrightarrow{\Phi} BU .$$

It is the fibration induced from the universal fibration

(**) $\qquad U \to EU \to BU$

by the map $1 - \psi^k : BU \to BU$. We consider the Serre spectral sequences of these fibrations. We know that

$$H^*(U; \mathbb{Z}) = \Lambda(y_1, y_2, \ldots)$$

with $\deg y_i = 2i - 1$.

Since BU is simply connected we have

$$E_2^{r,s} = H^r(BU; \mathbb{Z}) \otimes H^s(U; \mathbb{Z}) .$$

By [2] we know that the y_i's are transgressive in (**) with $d_{2i}y_i = \tilde{c}'_i \in E_{2i}^{2i,0}$. By naturality, the y_i are also transgressive in (*) with

$$d_{2i}y_i = (1 - \psi^k)\tilde{c}'_i = (1 - k^i)\tilde{c}'_i \in E_{2i}^{2i,0} .$$

By induction on i, we see that in (*) $E_{2i}^{*,*} = T_i \oplus L_i$ where T_i is a torsion group and L_i is $\mathbb{Z}$-free with basis the monomials $(\tilde{c}'_i)^{\alpha_i}(\tilde{c}'_{i+1})^{\alpha_{i+1}} \cdots y_i^{\beta_i} y_{i+1}^{\beta_{i+1}} \cdots$ with $\alpha_i \in \mathbb{N}$ and $\beta_i \in \{0, 1\}$. In particular, we have

$$E_{2i}^{2i,0} \simeq \mathbb{Z}\tilde{c}'_i \oplus T \quad \text{and} \quad E_\infty^{2i,0} = E_{2i+1}^{2i,0} \simeq \mathbb{Z}/s_i\tilde{c}'_i \oplus T ,$$

where T is a torsion group and $s_i = |k^i - 1|$. Since $E_\infty^{2i,0} = \Phi^* H^{2i}(BU; \mathbb{Z})$, this proves Proposition 8.1. □

Remark 8.2. This proof shows that $\tilde{c}_i$ generates a direct summand of order s_i in $\mathrm{Im}\, \Phi^*$ and that $\tilde{c}_{i-1}\tilde{c}_1$ is in $T \subset E_{2i}^{2i,0}$. Hence $\tilde{c}_i - \tilde{c}_{i-1}\tilde{c}_1 \in E_\infty^{2i,0} \subset \mathrm{Im}\, \Phi^*$ has order s_i.

Let G_∞ and SG_∞ be the direct limits $\varinjlim G_n$ and $\varinjlim \mathrm{SG}_n$ respectively. By performing the '+' construction on the classifying spaces we get the commutative diagram

$$\begin{tikzcd}
B\mathrm{SG}_n \arrow[r, "f"] \arrow[d] & B\mathrm{SG}_\infty \arrow[r] \arrow[d, "j"] & B\mathrm{SG}_\infty^+ \arrow[d] \\
BG_n \arrow[r, "g"] & BG_\infty \arrow[r] & BG_\infty^+ \sim JU(k)
\end{tikzcd}$$

where the maps are the obvious maps.

Since $H^2(B\mathrm{SG}_n; \mathbb{Z}/l) = 0$ for all $l \neq p$, we have $H^2(B\mathrm{SG}_n; \mathbb{Z}[\frac{1}{p}]) = 0$. This proves that $\tilde{c}_1 = 0$ in $H^2(B\mathrm{SG}_n; \mathbb{Z})$.

To prove the third part of Theorem 2.1, we observe that the map $BG_n \to JU(k)$

is a lifting of the map $BG_n \to BU$ obtained by the Brauer lifting of the canonical representation $BG_n \to BGL_n(\bar{\mathbb{F}}_p)$, where $\bar{\mathbb{F}}_p$ is an algebraic closure of $\mathbb{F}_p$.

Now if S is a l-Sylow subgroup of G_n or SG_n then l does not divide p and the Brauer character of the inclusion $S \to GL_n(\bar{\mathbb{F}}_p)$ is the character a complex representation of dimension n (cf. [6] and [8]). Hence $BS \to BG_n \to BU$ factorizes through $BU(n)$ and an argument of transfer gives the result.

The exact sequence $1 \to SG_\infty \to BG_\infty \xrightarrow{\det} G_1 \to 1$ splits and BG_∞^+ is an H-space (cf. [5]), so

$$BG_\infty^+ \quad \text{has the same homotopy type as} \quad BSG_\infty^+ \times G_1 \tag{11}$$

(cf. also [1]). Thus in mod l cohomology, j^* is surjective and its kernel is the ideal generated by e_1 and c_1.

By Theorems 1.1, 1.2 and 1.3, f^* and g^* are isomorphisms in mod l cohomology in degrees $\leq 2n$. Using the mapping cone of f and g, we see that f^* and g^* are injective in integral cohomology in these degrees away the p-torsion. But by [7] and [13], $H^*(BG_\infty; \mathbb{Z})$ has no p-torsion and then also $H^*(BSG_\infty; \mathbb{Z})$ by (11).

It remains to show that $j^*\tilde{c}_i$ has order $|k^i - 1|$. The monomorphism

$$h_n : G_{n-1} \to SG_n \subset G_n\,, \qquad A \mapsto \begin{pmatrix} \det A^{-1} & 0 \\ 0 & A \end{pmatrix}$$

gives a map

$$h : BG_\infty \to BSG_\infty \xrightarrow{j} BG_\infty\,.$$

The following lemma and Remark 8.2 give the result. □

Lemma 8.4. $h^*\tilde{c}_i = \tilde{c}_i - \tilde{c}_{i-1}\tilde{c}_1$.

Proof. The map $BG_n \to JU(k)$ is a lifting of the map $BG_n \to BU$ obtained by the Brauer lifting of the canonical representation $BG_n \to BGL_n(\bar{\mathbb{F}}_p)$, where $\bar{\mathbb{F}}_p$ is an algebraic closure of $\mathbb{F}_p$. Thus we can use the product formula to show that $h_n^*\tilde{c}_i = \tilde{c}_i - \tilde{c}_{i-1}\tilde{c}_1$ for $i < n$. The lemma follows from the injectivity of g^*. □

9. Proof of Theorem 2.3

The groups $\tilde{H}^*(\bigotimes_{i \in I} C_i)$ and $\tilde{H}^*(BG; \mathbb{Z}[\frac{1}{p}])$ are of finite type. So by using the mapping cone of h and the universal coefficient theorem it is enough to show that h^* is an isomorphism modulo l for all primes l distinct from p. Then we consider only the i's such that $i \equiv 0 \pmod r$. If $\tilde{x}$ is an integral cohomology class, we denote by x its reduction modulo l.

It is easily seen that the homomorphism $H^*(C_i \otimes \mathbb{Z}/l) \to H^*(K_i; \mathbb{Z}/l)$, induced by the inclusion, is injective. Let $y_i \in H^{2i-1}(K_i; \mathbb{Z}/l)$ be the image of the generator $\bar{b}_i \in H^{2i-1}(C_i \otimes \mathbb{Z}/l) \simeq \mathbb{Z}/l$. The monomials $x_i^\alpha y_i^\beta$, with $\alpha \in \mathbb{N}$ and $\beta \in \{0, 1\}$, form a basis of the image and we have $\beta y_i = \frac{s_i}{l} \tilde{x}_i$, where $\beta : H^{2i-1}(K_i; \mathbb{Z}/l) \to H^{2i}(K_i; \mathbb{Z})$ is the Bockstein homomorphism associated to the coefficient sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/l \to 0$. But β is injective, so by the choice of $\tilde{x}_i$ (cf. the proof of Lemma 2.2), y_i is homotopic to the map $K_i \to K(\mathbb{Z}/l, 2i-1)$ induced by the ring homomorphism $\mathbb{Z}/s_i \to \mathbb{Z}/l$. By the choice of γ_i we have then $\gamma_i^* y_i = e_i$, $\gamma_i^* \tilde{x}_i = \tilde{c}_i$ and $\gamma_i^* x_i = c_i$.

By identifying the cup products with the tensor products, we can regard $H^*(\otimes C_i \otimes \mathbb{Z}/l)$ as the subspace of $H^*(\prod K_i; \mathbb{Z}/l)$. The monomials

$$\omega = x_r^{\alpha_1} x_{2r}^{\alpha_2} \cdots x_{mr}^{\alpha_m} y_r^{\beta_1} y_{2r}^{\beta_2} \cdots y_{mr}^{\beta_m} ,$$

with $m = [n/r]$, form a basis of this subspace (if $G = \mathrm{SG}_n$ and $r = 1$, the basis consists of the monomials $x_2^{\alpha_2} \cdots x_n^{\alpha_n} y_2^{\beta_2} \cdots y_n^{\beta_n}$). But the basis of $H^*(G; \mathbb{Z}/l)$ described in Section 1 consists of the monomials $g^*\omega$. This proves Theorem 2.3. □

We prove Theorem 2.4 as above but we consider only the i's such that $i \equiv 0 \pmod{2d}$. □

Remark. For p odd we can obtain Theorem 2.4 as follows. The Adams operation ψ^q exists in sympletic K-theory for q odd and we have the fibration

$$J\mathrm{Sp}(q) \xrightarrow{\Phi} B\mathrm{Sp} \xrightarrow{1-\psi^q} B\mathrm{Sp} .$$

We know that $H^*(B\mathrm{Sp}; \mathbb{Z}) = \mathbb{Z}[\tilde{c}_2', \tilde{c}_4', \ldots]$ where $\tilde{c}_{2i}'$ is the image of $\tilde{c}_{2i}$ by the canonical map $B\mathrm{Sp} \to BU$. As before we show that $\Phi^* \tilde{c}_{2i}'$ has order $q^{2i} - 1$ in $H^{4i}(J\mathrm{Sp}(q); \mathbb{Z})$. By [7] there exists a commutative diagram

$$\begin{array}{ccc} B\mathrm{Sp}_\infty & \xrightarrow{\chi_q} & J\mathrm{Sp}(q) \\ \downarrow & & \downarrow \\ B\mathrm{GL}_\infty & \xrightarrow{\chi_q} & JU(q) \end{array}$$

where $\chi_q^+ : B\mathrm{Sp}_\infty^+ \to J\mathrm{Sp}(q)$ is an homotopy equivalence (cf. also [5]). So Theorem 1.4 gives the mod l cohomology of $J\mathrm{Sp}(q)$.

Since the diagram gives a "universal" class $\bar{e}_{2i} \in H^{4i-1}(J\mathrm{Sp}(q); \mathbb{Z}/q^{2i} - 1)$, we can prove Theorem 2.4 as we proved Theorem 2.3. Finally, theorem 2.4 describes the integral cohomology of $J\mathrm{Sp}(q)$ away the p-torsion, for p odd.

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